# Ergodic Theory and Measured Group Theory
## Lecture 8

**$L^p$-ergodic theorem.** Let $T$ be a pmp transformation on $(X,\mu)$. $\forall p \geq 1$, $\forall f \in L^p(X,\mu)$, $A_n^T f \to_{L^p} \underbrace{\mathbb{E}(f \mid \mathcal{B}_T)}_{=: \bar{f}}$ as $n \to \infty$.

**Proof.** Recall that we have $A_n^T f \to \bar{f}$ a.e. So if $f$ is bdd, then DCT implies that $A_n^T f \to_{L^p} \bar{f}$ as $n \to \infty$. For an arbitrary $f \in L^p$, let $f_k \to_{L^p} f$ s.t. each $f_k$ is bdd.

$$\| A_n^T f - \bar{f} \|_p \leq \overset{(1)}{\| A_n^T f - A_n^T f_k \|_p} + \overset{(2)}{\| A_n^T f_k - \bar{f}_k \|_p} + \overset{(3)}{\| \bar{f}_k - \bar{f} \|_p}$$

(1) $\| A_n^T f - A_n^T f_k \|_p = \| A_n^T (f - f_k) \|_p \leq \| f - f_k \|_p$ by Bridge$^+$ (b). $\| f - f_k \|_p \to 0$ as $k \to \infty$

(3) $\| \bar{f} - \bar{f}_k \|_p = \| \overline{(f - f_k)} \|_p \leq \| f - f_k \|_p$ (it's an $L^p$-contraction). $\| f - f_k \|_p \to 0$ as $k \to \infty$

(2) $\| A_n^T f_k - f_k \|_p \to 0$ as $n \to 0$ $f_k$ is bdd.

Hence $\| A_n^T - \bar{f} \|_p < \varepsilon$ $\forall \varepsilon > 0$. $\square$

**Ergodic decomposition.** Let $T$ be a pmp transformation on $(X,\mu)$. We'll try to understand how to partition $X$ into pieces on each of which $T$ is ergodic.

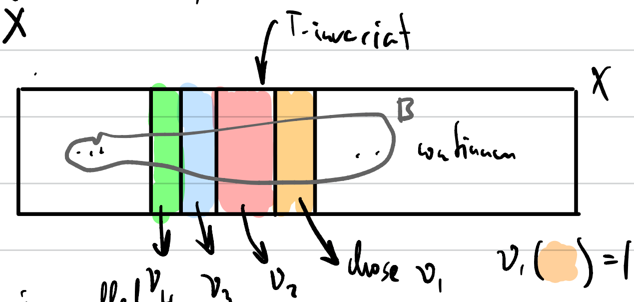
**Recall/Learn.** For a standard Borel space $X$, the space $P(X)$ of probability measures is standard Borel, where the Borel structure is induced by fixing a compact topology on $X$ and taking the weak*-topology on $P(X)$, i.e.

$$\mu_n \to \mu \quad :\iff \quad \forall f \in C(X), \quad \int_X f \, d\mu_n \to \int_X f \, d\mu .$$

**Ergodic decomposition (Farrel–Varadarajan).** Let $T$ be a pmp trans. on $(X,\mu)$. There is a $T$-invariant Borel $\nu : X \to P(X)$, $x \mapsto \nu_x$ s.t.

(i) Each $\nu_x$ is $T$-ergodic and $T$-invariant and $\nu_x(\nu^{-1}(\nu_x)) = 1$, i.e. for any $\lambda \in P(X)$ in the image of this map, $\lambda(\nu^{-1}(\lambda)) = 1$.

(ii) $\mu = \int_X \nu_x \, d\mu(x)$, i.e. $\forall B \subseteq X$ Borel, $\mu(B) = \int_X \nu_x(B) \, d\mu(x)$




The map $x \mapsto \nu_x$ is called the *ergodic decomposition* of $\mu$ over $T$.

**Cor.** If $x \mapsto \nu_x$ is the ergodic decomposition of $\mu$ over $T$, then $\forall L^1(X,\mu)$,

$$\mathbb{E}(f \mid \mathcal{B}_T) = \left(x \mapsto \int_X f \, d\nu_x\right).$$

In particular, $\forall$ Borel $B \subseteq X$,

$$\mathbb{E}(\mathbb{1}_B \mid \mathcal{B}_T) = \left(x \mapsto \nu_x(B)\right).$$

*Proof.* Exercise. $\boxed{\diagup}$

**Sketch of proof of the Erg. Dec. Thm.** Let $\mathcal{A}$ be a ctbl $T$-invariant algebra of Borel sets that generates all Borel sets. By $T$-invariant we mean $\forall A \in \mathcal{A}$, $T^{-1}(A) \in \mathcal{A}$. By the Carathéodory extension thm, each $\lambda \in P(X)$ is uniquely determined by its values on $\mathcal{A}$. Thus, we can identify $P(X)$ with a certain subset of $[0,1]^{\mathcal{A}}$. With a bit of care, we can make this subset closed by making $X$ compact and $T$ continuous. Define $\nu : X \to [0,1]^{\mathcal{A}}$ by $x \mapsto (\overline{\mathbb{1}}_A(x))_{A \in \mathcal{A}}$.

Hence $\forall A \in \mathcal{A}$,

$$\mu(A) = \int_X \overline{\mathbb{1}}_A(x)\, d\mu(x) = \int_X \nu_x(A)\, d\mu(x). \qquad (*)$$

Moreover, $\forall$ $T$-invariant $Y \subseteq X$,

$$\int_Y \mathbb{1}_A \, d\mu = \int_Y \overline{\mathbb{1}}_A(x)\, d(x) = \int_Y \nu_x(A)\, d(x). \qquad (**)$$

Because each $\nu_x$ is $T$-invariant and prob. measure on the algebra $\mathcal{A}$, it stays so on $\mathcal{B}(X)$. The rest of the properties of $\nu_x$ are proven using (**) and one also proves (*) for all Borel sets by an approximation argument. $\square$

## Ctbl groups and actions

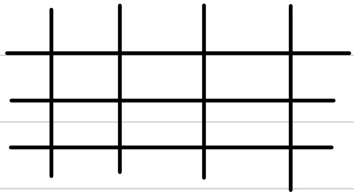
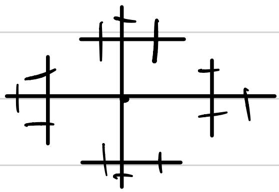
**Ctbl groups.**

- $\mathbb{Z}$, $\mathbb{Q}$
- Products of ctbl groups: $\mathbb{Z}^d$, $\mathbb{Z}^{<\infty} := \bigoplus_{n\in\mathbb{N}} \mathbb{Z} := \{x \in \mathbb{Z}^{\mathbb{N}} : \mathrm{supp}(x) \text{ is finite}\}$. Also $\mathbb{Z}^3 \times (\mathbb{Z}/2\mathbb{Z})$
- Free products: $\mathbb{F}^d := \underbrace{\mathbb{Z} * \mathbb{Z} * \dots * \mathbb{Z}}_{d}$, $T_d := \underbrace{\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \dots * \mathbb{Z}/2\mathbb{Z}}_{d}$.

**Def.** The *Cayley graph* of a group $\Gamma$ with a fixed generating set $S$ is the graph $\mathrm{Cay}(\Gamma, S)$ whose vertices are the elements of $\Gamma$ and $\forall\, \gamma, \delta \in \Gamma$,

$$\gamma \;\mathrm{Cay}(\Gamma, S)\; \delta \;:\Longleftrightarrow\; \gamma \cdot \sigma = \delta \text{ for some } \sigma \in S.$$

$\gamma \xrightarrow{s} \gamma s$ This is a labeled directed graph with labels $s \in S$.

- $\mathbb{Z}$  $S := \{\pm 1\}$.
- $\mathbb{Z}^2$  $S := \{(\pm 1, 0), (0, \pm 1)\}$.
- $\mathbb{F}_2$ 
- $T_3 := \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$